5.1 - Linear Models: Initial-Value Problems

Undamped motion:

$x(0) = -1$
$x'(0) = -2$

Example: A mass weighing 32 pounds stretches a spring 2 feet. Determine the amplitude and period of motion if the mass is initially released from a point 1 foot above the equilibrium position with an upward velocity of 2 ft/s. How many complete cycles will the mass have completed at the end of 4π seconds?

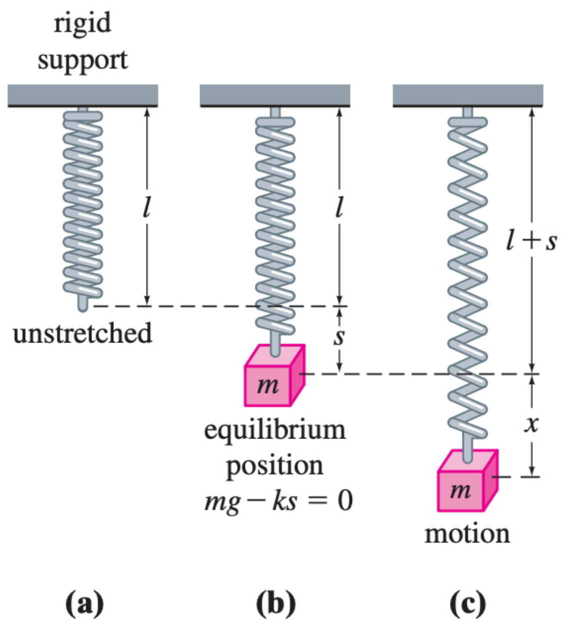


FIGURE 5.1.1 Spring/mass system

Hooke's law:

$F = k|s|$ $\quad F = 32$ lb

$32 = 2k$ $\quad |s| = 2$ ft

$k = 16$ lb/ft

(b) $mg = ks \Rightarrow mg - ks = 0$

force due to gravity and the mass

restoring force of spring

Assuming down is positive, if the mass is moving, we have $F = ma$ (Newton's 2nd law) with $a = \frac{d^2x}{dt^2}$

so $m\frac{d^2x}{dt^2} = -k(x+s) + mg = -kx \cancel{-ks+mg}$

Model: $m\frac{d^2x}{dt^2} + kx = 0$

$F = ma$

$F = 32$ lb ; $g = 32$ ft/s^2 $\quad 32 = m(32) \Rightarrow m = 1$ slug

Dividing by m and letting $\omega^2 = \frac{k}{m}$ yields

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \Rightarrow m^2 + \omega^2 = 0$$

$$m = \pm \omega i \quad \left(\omega = \sqrt{\tfrac{k}{m}}\right)$$

$$x(t) = c_1 \cos\omega t + c_2 \sin \omega t$$

Here, $k = 16$, $m = 1 \Rightarrow \omega = \sqrt{\frac{k}{m}} = 4$

$$x(t) = c_1 \cos 4t + c_2 \sin 4t$$

$x(0) = -1 \Rightarrow -1 = c_1$ $\qquad$ $x'(t) = -4c_1 \sin 4t + 4c_2 \cos 4t$

$x'(0) = -2 \Rightarrow -2 = 4c_2 \Rightarrow c_2 = -\frac{1}{2}$

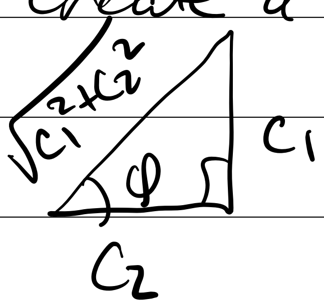
$$x(t) = -\cos 4t - \frac{1}{2}\sin 4t$$

period: $T = \frac{2\pi}{\omega} = \frac{2\pi}{4} = \frac{\pi}{2}\ s$

frequency $= \frac{1}{T} = \frac{2}{\pi}$ cycles/sec (Hz)

Consider $x(t) = c_1 \cos t + c_2 \sin t$

Create a right triangle w/ legs c_1, c_2.



$\sin\varphi = \frac{c_1}{\sqrt{c_1^2+c_2^2}}$, $\quad \cos\varphi = \frac{c_2}{\sqrt{c_1^2+c_2^2}}$

$$x(t) = \underbrace{\sqrt{c_1^2+c_2^2}}_{=A}\left(\boxed{\frac{c_1}{\sqrt{c_1^2+c_2^2}}}\cos t + \boxed{\frac{c_2}{\sqrt{c_1^2+c_2^2}}}\sin t\right)$$

$$x(t) = A\,(\sin\varphi \cos t + \cos\varphi \sin t)$$

$$x(t) = A \sin(t + \varphi)$$

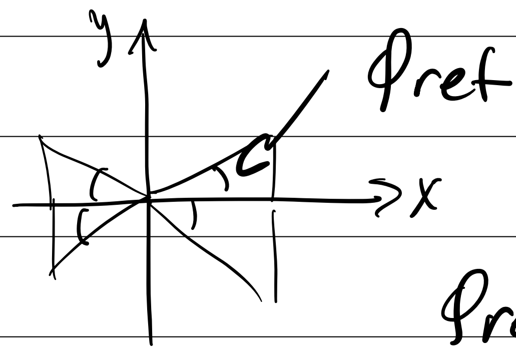
Here, $c_1 = -1$, $c_2 = -\frac{1}{2} \Rightarrow A = \sqrt{c_1^2 + c_2^2} = \frac{\sqrt{5}}{2}$

Amp: $\frac{\sqrt{5}}{2}$ ft

$$\frac{c_1}{A} = -\frac{2}{\sqrt{5}} = \sin\phi \qquad \Rightarrow \phi \in Q\,III$$

$$\frac{c_2}{A} = -\frac{1}{\sqrt{5}} = \cos\phi$$

using $\sin\phi = \frac{2}{\sqrt{5}}$, we'll find ϕ_{ref} in $Q\,I$



$$\phi_{ref} = \sin^{-1}\left(\frac{2}{\sqrt{5}}\right)$$

$$\approx 63^\circ \approx 1.107$$

$$\phi_{ref} + \pi \approx 4.25 \quad (\text{in } Q\,III)$$

$$x(t) = \frac{\sqrt{5}}{2} \sin(x + 4.25)$$

Recap: $\omega^2 = k/m \Rightarrow \omega = \sqrt{\frac{k}{m}}$

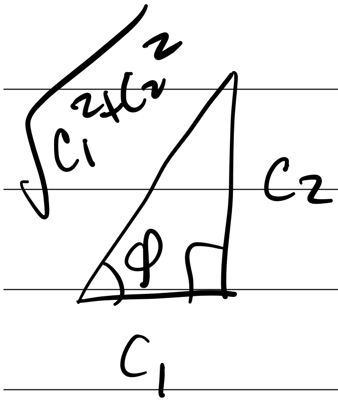
$$x(t) = c_1 \cos\omega t + c_2 \sin\omega t$$

per: $\frac{2\pi}{\omega}$ s, freq: $\frac{\omega}{2\pi}$ Hz

amp: $\sqrt{c_1^2 + c_2^2}$, $\phi_{ref} = \sin^{-1} \frac{c_1}{\sqrt{c_1^2 + c_2^2}}$

OR $\tan\phi = \frac{c_1}{c_2}$

If we choose to work with the cosine,



$$x(t) = \sqrt{c_1^2+c_2^2}\left(\frac{c_1}{\sqrt{c_1^2+c_2^2}}\cos\omega t + \frac{c_2}{\sqrt{c_1^2+c_2^2}}\sin\omega t\right)$$

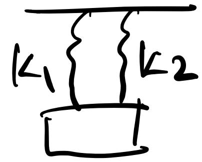
$$\cos\varphi = \frac{c_1}{c_1^2+c_2^2} \qquad \sin\varphi = \frac{c_2}{\sqrt{c_1^2+c_2^2}}$$

$$x(t) = A\left(\cos\varphi\cos\omega t + \sin\varphi\sin\omega t\right)$$

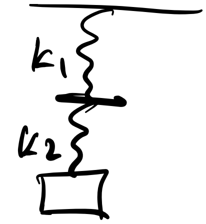
$$x(t) = A\cos(\omega t - \varphi)$$

$$\tan\varphi = \frac{c_2}{c_1}$$

Multiple springs in parallel: $k_{\text{eff}} = k_1 + k_2$



Multiple springs in series: $k_{\text{eff}} = \dfrac{k_1 k_2}{k_1 + k_2}$



Damped motion: $m\dfrac{d^2x}{dt^2} = -kx - \beta\dfrac{dx}{dt}$

damping constant

ma

restoring force from spring

instantaneous velocity

→ damping force is proportional to instantaneous velocity

mass

$$m\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + kx = 0$$

Let $2\lambda = \dfrac{\beta}{m}$ and $\omega^2 = \dfrac{k}{m}$

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = 0$$

Aux equation roots: $m = \dfrac{-2\lambda \pm \sqrt{4\lambda^2 - 4\omega^2}}{2}$

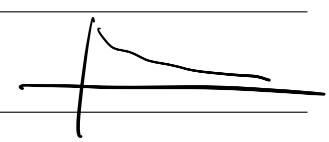
$$m = -\lambda \pm \sqrt{\lambda^2 - \omega^2}$$

Case I: $\lambda^2 - \omega^2 > 0$

$$x(t) = c_1 e^{(-\lambda + \sqrt{\lambda^2 - \omega^2})t} + c_2 e^{(-\lambda - \sqrt{\lambda^2 - \omega^2})t}$$

This does not oscillate.

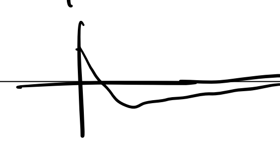
This system is overdamped



Case II: $\lambda^2 - \omega^2 = 0$

Not gonna happen by accident

$$x(t) = c_1 e^{-\lambda t} + c_2 t e^{-\lambda t}$$



This is critically damped.

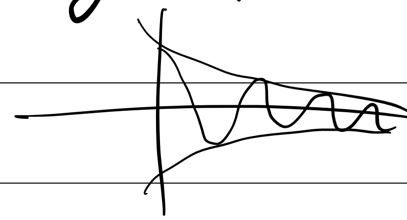
The mass can cross equilibrium at most one time for Case I and Case II.

<u>Case III</u> $\lambda^2 - \omega^2 < 0$

$$x(t) = e^{-\lambda t}\left[c_1 \cos(\omega^2-\lambda^2)t + c_2 \sin(\omega^2-\lambda^2)t\right]$$

This oscillates with decreasing amplitude.

This is <u>underdamped</u>.



Driven motion: $m\dfrac{d^2x}{dt^2} + kx + \beta\dfrac{dx}{dt} = f(t)$

underdetermined coefficients or variation of parameters

LRC-series circuit: $L\dfrac{di}{dt} + Ri + \dfrac{1}{C}q = E(t)$, which is

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t)$$

$$i = \frac{dq}{dt}$$

$$\frac{di}{dt} = \frac{d^2q}{dt^2}$$

LR-series circuit:

$$L\frac{di}{dt} + \boxed{Ri} = E(t)$$

RC-series circuit:

$$\boxed{R\frac{dq}{dt}} + \frac{1}{C}q = E(t)$$

underdamped, critically damped, overdamped have the same meanings as ~~for~~ spring systems.

The complementary function is the transient term/solution.

The particular solution is the steady-state term/solution.

Example: Find the charge on the capacitor in an LRC-series circuit when $L = \frac{1}{4}$ h, $R = 20\ \Omega$, $C = \frac{1}{300}$ f, $E(t) = 0$ V, $q(0) = 4$ C, and $i(0) = 0$ A. Is the charge on the capacitor ever equal to zero?

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t)$$

$$\frac{1}{4}\frac{d^2q}{dt^2} + 20\frac{dq}{dt} + 300q = 0$$

$$\ddot{q} + 80\dot{q} + 1200q = 0$$

(Note: $\dot{q} = \frac{dq}{dt}$, a derivative with respect to time.)

$$(m+60)(m+20) = 0$$

$$m = -20, -60$$

$$q(t) = c_1 e^{-20t} + c_2 e^{-60t}$$

$$q'(t) = i(t) = -20c_1 e^{-20t} - 60c_2 e^{-60t}$$

$q(0) = 4 \Rightarrow 4 = c_1 + c_2$

$i(0) = 0 \Rightarrow 0 = -20c_1 - 60c_2$

$\rightarrow$

$$80 = 20c_1 + 20c_2$$
$$0 = -20c_1 - 60c_2$$

$c_1 = 6 \longleftarrow c_2 = -2$

$$q(t) = 6e^{-20t} - 2e^{-60t}$$

Is charge ever zero? (Does $q = 0$ have a solution?)

$$0 = 6e^{-20t} - 2e^{-60t} \Rightarrow e^{40t} = \frac{1}{3}$$

$t = \frac{1}{40}\ln\frac{1}{3} \leftarrow$ this is negative.

The charge is never zero for $t > 0$.